

简单的答案解析

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一

1. $\begin{cases} 1, & t \text{ 为偶数且 } k=1 \\ 0, & \text{else} \end{cases}; \quad \text{非平稳}$

2. $\gamma_k(X_t, X_{t+k}) = \begin{cases} 6\sigma^2, & k=0 \\ 2\sigma^2, & |k|=1 \\ 0, & |k|>1 \end{cases}; \quad \text{Corr}(X_t, X_{t+k}) = 0$

3. B

4. $-1 < a < 0.5$

5. $\frac{\alpha_0}{1-\alpha_1}$

6. 5.75

7. $4\pi^2 f^3(\lambda)$

8. 白噪声; χ^2 分布或者正态分布; $\{x_t\}$ 是独立白噪声。

9. $\frac{1}{1-\frac{1}{2}\phi_1-\dots-(\frac{1}{2})^p\phi_p}$

10. 8/21, 1/21

二

(1). 当 $k=0$,

$$\gamma(0) = \int_{-\pi}^{\pi} \frac{\sigma^2(\pi - |\lambda|)}{\pi^2} d\lambda = 2\sigma^2 - \sigma^2 = \sigma^2.$$

当 $k \neq 0$

$$\gamma(k) = \int_{-\pi}^{\pi} \cos k\lambda f(\lambda) d\lambda = \frac{2\sigma^2}{k^2\pi^2} [1 - \cos k\pi] = \frac{2\sigma^2}{k^2\pi^2} [1 - (-1)^k].$$

(2). 由 Levinson 递推公式, 我们有

$$\nu_0 = \gamma(0), \theta_{1,1} = \gamma(1)/\gamma(0), \nu_1 = \gamma(0) - \gamma^2(1)/\gamma(0) \text{ and } \theta_{2,1} = (\gamma(1) - \theta_{1,1}\theta_{2,2}\nu_0)/\nu_1, \theta_{2,2} = \gamma(2)/\nu_0.$$

最后, 我们可以得到 $\theta_{2,1} = 4\pi^2/(\pi^4 - 16)$, $\theta_{2,2} = 0$.

三

(1).

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \quad EX_t \varepsilon_{t-k} = \begin{cases} \sigma^2 \psi_k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$\psi_j = a_1 \psi_{j-1} + b_j \Rightarrow \psi_0 = 1, \psi_1 = 1.5, \psi_j = \frac{9}{5} \left(\frac{4}{5}\right)^{j-2}, j \geq 2$$

$$\gamma_0 = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 = 12.25\sigma^2$$

$$X_t = 0.8X_{t-1} + \varepsilon_t + 0.7\varepsilon_{t-1} + 0.6\varepsilon_{t-2}$$

两边同乘 X_{t-1} , 取期望

$$\gamma_1 = 0.8\gamma_0 + (0.7\psi_0 + 0.6\psi_1)\sigma^2 \Rightarrow \rho_1 = 0.8 + (0.7\psi_0 + 0.6\psi_1)/12.25$$

两边同乘 X_{t-2} , 取期望 $\gamma_2 = 0.8\gamma_1 + 0.6\psi_1\sigma^2 \Rightarrow \rho_2 = 0.8\rho_1 + 0.6\psi_1/12.25$

两边同乘 $X_{t-k}, k \geq 3$, 取期望 $\gamma_k = 0.8\gamma_{k-1} \Rightarrow \rho_k = 0.8\rho_{k-1}, k \geq 3$

(2).

$$A(z) = 1 - 0.8z, B(z) = 1 + 0.7z + 0.6z^2$$

$$A(z) = 0 \Rightarrow z = 1.25, B(z) = 0 \Rightarrow |z_1| = |z_2| = \sqrt{\frac{5}{3}}$$

序列是因果可逆的。

(3).

$$\varepsilon_t = \frac{A(L)}{B(L)} X_t$$

$$\text{设 } \frac{A(z)}{B(z)} = \sum_{j=0}^{\infty} \xi_j z^j$$

$$(1 + b_1 z + b_2 z^2) \sum_{j=0}^{\infty} \xi_j z^j = 1 - a_1 z$$

$$\Rightarrow \xi_0 + (\xi_1 + b_1 \xi_0) z + \sum_{j=2}^{\infty} (\xi_j + b_1 \xi_{j-1} + b_2 \xi_{j-2}) z^j = 1 - a_1 z$$

$$\Rightarrow \xi_0 = 1, \xi_1 = -a_1 - b_1 \xi_0 = -1.5, \xi_2 = 0.45, \xi_j = -b_1 \xi_{j-1} - b_2 \xi_{j-2}, j \geq 2$$

记 α, β 为 $B(\frac{1}{z}) = 0$ 的两个根, 令 $c_1 = \frac{\beta \xi_1 - \xi_2}{\beta - \alpha} = \frac{0.45 + 1.5\beta}{\alpha - \beta}, c_2 = \frac{\xi_2 - \alpha \xi_1}{\beta - \alpha} = \frac{0.45 + 1.5\alpha}{\beta - \alpha}$, 则

$$\xi_j = c_1 \alpha^{j-1} + c_2 \beta^{j-1}, j \geq 3$$

$$\varepsilon_t = \sum_{j=0}^{\infty} \xi_j X_{t-j} = X_t - 1.5X_{t-1} + 0.45X_{t-2} + \sum_{j=3}^{\infty} (c_1 \alpha^{j-1} + c_2 \beta^{j-1}) X_{t-j}$$

四

(1). 最佳线性预测 $\hat{X}_t(l), l = 1, 2, \dots$ 由下式确定:

$$\begin{cases} \hat{X}_t(l) = \frac{1}{16} \hat{X}_t(l-2), & l = 1, 2, \dots, \\ \hat{X}_t(-1) = X_{t-1}, \hat{X}_t(0) = X_t. \end{cases}$$

由于特征方程

$$\lambda^2 - \frac{1}{16} = 0$$

有相异实根

$$\lambda_1 = \frac{1}{4}, \quad \lambda_2 = -\frac{1}{4},$$

因此

$$\hat{X}_t(l) = \frac{c_1}{4^l} + \frac{c_2}{(-4)^l}, \quad l = -1, 0, 1, \dots,$$

其中 c_1, c_2 由下式确定:

$$X_t = c_1 + c_2, \quad X_{t-1} = 4(c_1 - c_2),$$

即

$$c_1 = \frac{X_t}{2} + \frac{X_{t-1}}{8}, \quad c_2 = \frac{X_t}{2} - \frac{X_{t-1}}{8}.$$

因而

$$\begin{aligned} \hat{X}_t(l) &= \frac{1}{4^l} \left(\frac{X_t}{2} + \frac{X_{t-1}}{8} \right) + \frac{1}{(-4)^l} \left(\frac{X_t}{2} - \frac{X_{t-1}}{8} \right) \\ &= \frac{1 + (-1)^l}{2^{2l+1}} X_t + \frac{1 - (-1)^l}{2^{2l+3}} X_{t-1}, \quad l = -1, 0, 1, \dots \end{aligned}$$

(2). 由 (1) 可得

$$\begin{aligned} e_t(l) &= X_{t+l} - \hat{X}_t(l) \\ &= X_{t+l} - \frac{1 + (-1)^l}{2^{2l+1}} X_t - \frac{1 - (-1)^l}{2^{2l+3}} X_{t-1}, \quad l = -1, 0, 1, \dots \end{aligned}$$

因而

$$\begin{aligned} E[e_t(l)^2] &= E \left\{ \left[X_{t+l} - \frac{1 + (-1)^l}{2^{2l+1}} X_t - \frac{1 - (-1)^l}{2^{2l+3}} X_{t-1} \right]^2 \right\} \\ &= E(X_{t+l}^2) + \left[\frac{1 + (-1)^l}{2^{2l+1}} \right]^2 E(X_t^2) + \left[\frac{1 - (-1)^l}{2^{2l+3}} \right]^2 E(X_{t-1}^2) \\ &\quad - \frac{1 + (-1)^l}{2^{2l}} E(X_{t+l} X_t) - \frac{1 - (-1)^l}{2^{2(l+1)}} E(X_{t+l} X_{t-1}) \\ &= \left\{ 1 + \frac{1 + (-1)^l}{2} \frac{1}{2^{4l}} + \frac{1 - (-1)^l}{2} \frac{1}{2^{4(l+1)}} \right\} \gamma(0) - \frac{1 + (-1)^l}{2^{2l}} \gamma(l) - \frac{1 - (-1)^l}{2^{2(l+1)}} \gamma(l+1), l = 1, 2, \dots \end{aligned}$$

(3). 因为 $|\gamma(k)| \leq \gamma(0)$, 由 (2) 可得

$$E[e_t(l)^2] \rightarrow \gamma(0), \quad l \rightarrow +\infty.$$

五

(1) AR(p) 模型的谱密度函数为:

$$f_X(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-ik\omega}.$$

按照上面思路求出自相关函数就可以。

此外, 书本上 p67 有推论 3.2, AR(p) 有谱密度:

$$f_X(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-ik\omega} = \frac{\sigma^2}{2\pi |A(e^{i\omega})|^2}.$$

在这道题目中, $A(z) = 1 - 0.9z^3$.

结果为

$$f_X(\omega) = \frac{\sigma^2}{2\pi(1 - 0.9e^{3i\omega})^2}.$$

(2) 对一个绝对可和的线性滤波器而言, 其谱密度为

$$f_Y(\lambda) = |H(e^{-i\lambda})|^2 f_X(\lambda),$$

其中 $H(z) = (z^{-1} + 1 + z)/3$ 。将上面的结果带入就可以了。

结果为

$$f_Y(\omega) = \frac{(e^{i\omega} + 1 + e^{-i\omega})^2}{9} f_X(\omega) = \frac{\sigma^2(e^{i\omega} + 1 + e^{-i\omega})^2}{18\pi(1 - 0.9e^{3i\omega})^2}.$$

再利用公式 $e^{iw} = \cos w + i \sin w$, 求得 $e^{3iw}|_{w=2\pi/3} = 1$ 和 $(e^{i\omega} + 1 + e^{-i\omega})|_{w=2\pi/3} = 0$.